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W_v PATHS IN THE PROJECTIVE PLANE

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If a graph G is embedded in a manifold then a path in G is said to be a W_v path if and only if it never returns to a face of the graph once it leaves it. We prove that between each two vertices of a cell complex in the projective plane there is a W_v path.

1. Introduction

A well-known conjecture in the field of convex polytopes is that, given any two vertices of a convex polytope, there is a path of vertices and edges joining them such that once the path leaves a face of the polytope it never returns to it. Such a path is called a *W_v-path*. A proof of this W_v conjecture would settle the famous Hirsch Conjecture that states that any two vertices of a d-dimensional convex polytope can be joined by a path with at most n-d edges, where n is the number of $(d$-$1)$-dimensional faces of the polytope.

Klee conjectured that the W_v conjecture might be true for cell complexes more general than the boundary complexes of convex polytopes [3]. David Larman [4] has shown that for a very general type of 2-dimensional complex, the conjecture is false. Mani and Walkup [6] have shown that the conjecture is false for 3-dimensional spheres. The only positive result is that the conjecture is true for d-polytopes for $0 \leqslant d \leqslant 3$. In this paper we show that it is true for cell complexes that are homeomorphic to the projective plane.

2. Definitions and notation

The W_v conjecture is known to be true for convex 3-dimensional polytopes (see [2, Ch. 16]). By a theorem of Steinitz [5] this is equivalent to saying that the conjecture is true for 3-connected graphs on the sphere. An important property that 3-connected graphs on the sphere have is that given any two faces, they intersect on a single edge, a single vertex, or not at all. When two faces meet in this way we say that they meet *properly*. This is the property that we shall use to generalize planar 3-connected graphs to graphs in other 2-manifolds. (Hereafter the term *manifold* will be used for a 2-dimensional manifold).

Let G be a graph embedded in a manifold. The manifold minus the graph consists of various connected components. The closures of these components will

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be called the *faces* of the graph. If all the faces are closed cells and all faces meet properly and each vertex has valence at least 3, then we call the graph a *polyhedral map* in the manifold. A theorem by the author [1] gives us that all polyhedral maps are 3-connected; that is, given any two vertices, there are three paths joining them meeting only at their endpoints. Such a set of paths is called an *independent set*.

For any path, its *length* is defined to be the number of edges of the path. If x and y are two vertices of a path, then the portion of the path joining x and y is denoted $P[x, y]$. The path $P[x, y]$ minus its endpoints is denoted $P(x, y)$. The *distance* from x to y on P is defined to be the length of $P[x, y]$. Any vertex of a path from u to v other than u or v will be called an *interior* vertex of the path.

We say that a circuit in a graph in a manifold *bounds*, provided it bounds a cell that is a subset of the manifold.

If P is a path in a polyhedral map we define a *revisit* of P to a face F to be a pair of vertices (x, y) such that $P[x, y] \cap F = \{x, y\}$. We shall say that the revisit *involves* x and y. If the path is from u to v then we say that (x, y) is a *first revisit* if there is no revisit (z, w) with z closer to u than x on the path P and no revisit (x, w) with w closer to v than y on P. A path is a W_v path if it has no revisits.

Let (x, y) be a revisit of a path P to a face F. The two paths along F from x to y will be denoted $F[x, y]$ and $F^*[x, y]$. The revisit (x, y) is called *planar* provided either $F[x, y] \cup P[x, y]$ or $F^*[x, y] \cup P[x, y]$ is a circuit that bounds. (Note that if one bounds then so does the other.) If these two circuits bound then one of them bounds a cell $C(x, y)$ that does not contain the face F. Whenever a revisit (x, y) is planar we shall assume that the paths along F are labeled so that $C(x, y)$ is bounded by $F^*[x, y] \cup P[x, y]$.

3. W_v paths in the projective plane

Lemma 1. *If u and v are vertices of a face F of a polyhedral map in a manifold, then there is a W_v path from u to v.*

Proof. If u and v are endpoints of an edge, that edge is a W_v path. If u and v do not lie on a common edge then any path from u to v on F is a W_v path. □

Lemma 2. *If P, P' and P'' are independent paths joining u and v in a polyhedral map M in a manifold, and if the union of each two paths is a circuit that bounds, then there is a path from u to v such that every revisit is planar.*

Proof. Suppose that P'' is not in the cell C bounded by $P \cup P'$. In this case, either the cell C' bounded by $P' \cup P''$ meets C on P', or the cell C'' bounded by $P \cup P''$ meets C on P. In either case one of the three paths lies in a cell bounded by the other two. Let us suppose that P lies in the cell C' bounded by $P' \cup P''$. Any face

F revisited by P must involve an interior vertex of P or else we are done by Lemma 1. If F meets interior vertices of P then the face lies in the cell C' and the revisit must be planar. □

If (x, y) is a planar revisit of a path $P[u, v]$ to a face F we say that (x, y) is of *Type* 1 if u and v are not in the interior of $C(x, y)$. If u and v are both in the interior of C we say (x, y) is of *Type* 2. If v is in the interior of C and u is not, (x, y) is of *Type* 3; and if u is in the interior of C and v is not (x, y) is of *Type* 4.

If (x, y) is a revisit of a path P to a face F, we define $\bar{x}$ and $\bar{y}$ to be vertices such that $P(\bar{x}, x)$ and $P(y, \bar{y})$ are maximal paths on F ending at x and y respectively. Note that is possible for $x = \bar{x}$ or $y = \bar{y}$. By a *modification* of P we mean a replacement of part of P by a path along F as follows:

1) $P(x, y)$ is replaced by xy if xy is an edge, otherwise
2) $P(x, y)$ is replaced by $F^*(x, y)$ if (x, y) is of Type 1;
3) $P(x, y)$ is replaced by $F(x, y)$ if (x, y) is of Type 2;
4) $P(x, \bar{y})$ is replaced by $F^*(x, \bar{y})$ if (x, y) is of Type 3;
5) $P(\bar{x}, y)$ is replaced by $F^*(\bar{x}, y)$ if (x, y) is of Type 4.

Lemma 3. *Let $P(u, v)$ be a path in a polyhedral map such that all revisits are planar and no revisits are of Type 4. If repeated modifications applied to first revisits never create revisits of Type 4, then repeated modifications will transform P into a W_v path from u to v.*

Proof. We will be taking a path P with no nonplanar revisits and applying a sequence of modifications to first revisits eventually producing a W_v path. It can happen that a first revisit (x, y) is not unique. In this case, where two faces F and F' are revisited at x and y, the vertices x and y must be joined by an edge lying on both F and F'. If $P(x, y)$ is replaced by the edge xy it is easily seen that the revisits of the new path are a subset of the set of revisits of the old path. Therefore, anytime we have produced a path with a nonunique first revisit we shall perform this replacement by xy and then proceed.

Clearly, modifications of Type 1 do not create new revisits. For modifications of Types 2 and 3, any new revisit must have its vertices on the path that replaces $P(x, y)$, but this is a path along F and cannot revisit a face, unless xy is an edge. Thus no new revisits are created, and modifications of Types 2 and 3 decrease the number of revisits.

Suppose (s, t) is a revisit to face F' created by a modification of Type 4. The face F' lies inside $C(x, y)$ thus (s, t) cannot be of Type 4. Since F' is in $C(x, y)$ with $t \in P(\bar{y}, v]$ the distance of v to the second vertex of the first revisit is decreased by modifications of Type 4, as well as with Types 1, 2, and 3.

Repeated applications of these modifications must therefore eventually stop. This occurs when a W_v path is produced. □

Theorem 1. *If P is a path from u to v in a polyhedral map and if P has no nonplanar revisits, then repeated modifications can be applied to P to produce a W_v path from u to v.*

Proof. As the proof of Lemma 3 shows, repeated modifications using first revisits will decrease the distance from v to the second vertex of the first revisit as long as no modification of Type 5 is performed. Suppose at some stage in applying modifications the path has been modified so that the first revisit (x, y) of the resulting path P' to a face F is of Type 4. If no new revisit is a revisit to a face lying in $C(x, y)$ then the distance of v to the last vertex of the first revisit is decreased.

Suppose the new first revisit to P'' is a revisit (s, t) to a face F', lying in $C(x, y)$. When modifications are now applied using revisits to the portion of P'' lying in C, new revisits of Type 4 cannot be created (see Fig. 1). Using Lemma 3 and the symmetric roles played by u and v we have that repeated modifications applied to $P''(u, y)$ (regarded as a path from y to u) will produce a W_v path from u to y. This path together with $P[y, u]$, can only have revisits that involve vertices of $P[y, v]$, and we have again decreased the distance from v to the last vertex of the first revisit. The process must stop but can only stop when there are no revisits. □

Theorem 2. *Each two vertices u and v of a polyhedral map M in the projective plane can be joined by a W_v path.*

Proof. Since M is 3-connected there are three independent paths P, P' and P'' joining u and v. If each two of these paths form a circuit that bounds, then by Lemma 2 we are done. We assume that $P \cup P'$ is a circuit that does not bound. We cut the projective plane along this circuit and obtain a cell C with paths R and R' on the boundary of C corresponding to P and paths Q and Q' on the boundary of C corresponding to P' (see Fig. 2).

The path P'' lies in C and must have a nonplanar revisit to some face F, for otherwise we are done by Lemma 3. Let such a revisit be (a, b). In the cell C, the circuit $F[a, b] \cup P''[a, b]$ bounds a cell, unless a or b is u or v, as shown in Fig. 3. If neither a nor b is u or v, then after identifications are made on the boundary of the cell to produce the projective plane, this circuit will still bound a cell and thus the revisit is planar. We may assume therefore that the revisit to F involves u and some interior vertex a of P''.

There must be a face F' with a nonplanar revisit by P or we are done by Lemma 3. In C such a face would meet R and R'. In C the path $P''[u, a] \cup F[a, u]$ separates all interior vertices of R from all vertices of R' except u, thus F' must meet u and an interior vertex of R (note that it cannot meet both u and v or we would be done by Lemma 1). This is illustrated in Fig. 4. Now there is no way for a face to meet both Q and Q' without meeting both u and v, thus P' has no nonplanar revisits. By Lemma 3 we are now done. □

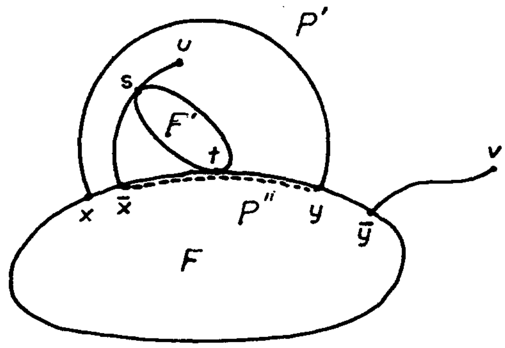

Fig. 1

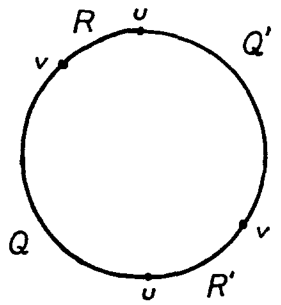

Fig. 2

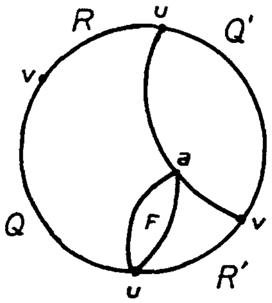

Fig. 3

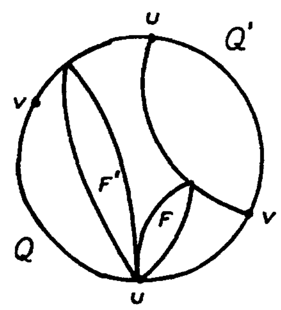

Fig. 4

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